

# Algorithm for generating new explicitly solvable Schrödinger type equations

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**Abstract.** In this note we present an algorithm to generate new Schrödinger type equations explicitly solvable in terms of orthogonal polynomials or associated special functions.

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## 1. Introduction

Many problems in quantum mechanics and mathematical physics lead to equations of the type

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \quad (1)$$

where  $\sigma(s)$  and  $\tau(s)$  are polynomials of at most second and first degree, respectively, and  $\lambda$  is a constant. These equations are usually called *equations of hypergeometric type* [10], and each of them can be reduced to the self-adjoint form

$$[\sigma(s)\varrho(s)y'(s)]' + \lambda\varrho(s)y(s) = 0 \quad (2)$$

by choosing a function  $\varrho$  such that  $(\sigma\varrho)' = \tau\varrho$ .

The equation (1) is usually considered on an interval  $(a, b)$ , chosen such that

$$\begin{aligned} \sigma(s) &> 0 & \text{for all } s &\in (a, b) \\ \varrho(s) &> 0 & \text{for all } s &\in (a, b) \\ \lim_{s \rightarrow a} \sigma(s)\varrho(s) &= \lim_{s \rightarrow b} \sigma(s)\varrho(s) = 0. \end{aligned} \quad (3)$$

Since the form of the equation (1) is invariant under a change of variable  $s \mapsto cs + d$ , it is sufficient to analyse the cases presented in table 1. Some restrictions are to be imposed to  $\alpha, \beta$  in order the interval  $(a, b)$  to exist.

**Table 1.** The main cases

$\sigma(s)$	$\tau(s)$	$\varrho(s)$	$\alpha, \beta$	$(a, b)$
1	$\alpha s + \beta$	$e^{\alpha s^2/2 + \beta s}$	$\alpha < 0$	$\mathbb{R}$
$s$	$\alpha s + \beta$	$s^{\beta-1}e^{\alpha s}$	$\alpha < 0, \beta > 0$	$(0, \infty)$
$1 - s^2$	$\alpha s + \beta$	$(1+s)^{-(\alpha-\beta)/2-1}(1-s)^{-(\alpha+\beta)/2-1}$	$\alpha < \beta < -\alpha$	$(-1, 1)$
$s^2 - 1$	$\alpha s + \beta$	$(s+1)^{(\alpha-\beta)/2-1}(s-1)^{(\alpha+\beta)/2-1}$	$-\beta < \alpha < 0$	$(1, \infty)$
$s^2$	$\alpha s + \beta$	$s^{\alpha-2}e^{-\beta/s}$	$\alpha < 0, \beta > 0$	$(0, \infty)$
$s^2 + 1$	$\alpha s + \beta$	$(1+s^2)^{\alpha/2-1}e^{\beta \arctan s}$	$\alpha < 0$	$\mathbb{R}$

The equation (1) defines an infinite sequence of orthogonal polynomials in the case  $\sigma(s) \in \{1, s, 1 - s^2\}$ , and a finite one in the case  $\sigma(s) \in \{s^2 - 1, s^2, s^2 + 1\}$ . The literature discussing special function theory and its application to mathematical and theoretical physics is vast, and there are a multitude of different conventions concerning the definition of functions. The table 1 allows one to pass in each case from our parameters  $\alpha, \beta$  to the parameters used in different approach.

In section 2 we briefly present some results concerning orthogonal polynomials, associated special functions, hypergeometric type operators and related Schrödinger type operators which are needed in section 3. In quantum mechanics there exist potentials, called quasi-exactly solvable, for which it is possible to find a finite portion of the energy spectrum and the associated eigenfunctions exactly and in closed form [1, 6, 7, 8, 11, 12]. An algorithm for generating new explicitly solvable systems and some applications are presented in section 3.

## 2. Orthogonal polynomials and associated special functions

Let  $\tau(s) = \alpha s + \beta$  be a fixed polynomial, and let

$$\lambda_\ell = -\frac{\sigma''(s)}{2}\ell(\ell-1) - \tau'(s)\ell = -\frac{\sigma''}{2}\ell(\ell-1) - \alpha\ell \quad (4)$$

for any  $\ell \in \mathbb{N}$ . It is well-known [10] that for  $\lambda = \lambda_\ell$ , the equation (1) admits a polynomial solution  $\Phi_\ell = \Phi_\ell^{(\alpha, \beta)}$  of at most  $\ell$  degree

$$\sigma(s)\Phi_\ell'' + \tau(s)\Phi_\ell' + \lambda_\ell\Phi_\ell = 0. \quad (5)$$

If the degree of the polynomial  $\Phi_\ell$  is  $\ell$  then it satisfies the Rodrigues formula [10]

$$\Phi_\ell(s) = \frac{B_\ell}{\varrho(s)} \frac{d^\ell}{ds^\ell} [\sigma^\ell(s) \varrho(s)] \quad (6)$$

where  $B_\ell$  is a constant. Based on the relation

$$\begin{aligned} & \{ \delta \in \mathbb{R} \mid \lim_{s \rightarrow a} \sigma(s) \varrho(s) s^\delta = \lim_{s \rightarrow b} \sigma(s) \varrho(s) s^\delta = 0 \} \\ &= \begin{cases} [0, \infty) & \text{if } \sigma(s) \in \{1, s, 1-s^2\} \\ [0, -\alpha) & \text{if } \sigma(s) \in \{s^2-1, s^2, s^2+1\} \end{cases} \end{aligned} \quad (7)$$

one can prove [2, 3, 10] that the system of polynomials  $\{\Phi_\ell \mid \ell < \Lambda\}$ , where

$$\Lambda = \begin{cases} \infty & \text{for } \sigma(s) \in \{1, s, 1-s^2\} \\ \frac{1-\alpha}{2} & \text{for } \sigma(s) \in \{s^2-1, s^2, s^2+1\} \end{cases} \quad (8)$$

is orthogonal with weight function  $\varrho(s)$  in  $(a, b)$ . This means that equation (1) defines an infinite sequence of orthogonal polynomials

$$\Phi_0, \Phi_1, \Phi_2, \dots$$

in the case  $\sigma(s) \in \{1, s, 1-s^2\}$ , and a finite one

$$\Phi_0, \Phi_1, \dots, \Phi_L$$

with  $L = \max\{\ell \in \mathbb{N} \mid \ell < (1-\alpha)/2\}$  in the case  $\sigma(s) \in \{s^2-1, s^2, s^2+1\}$ .

The polynomials  $\Phi_\ell^{(\alpha, \beta)}$  can be expressed (up to a multiplicative constant) in terms of the classical orthogonal polynomials as [3]

$$\Phi_\ell^{(\alpha, \beta)}(s) = \begin{cases} H_\ell \left( \sqrt{\frac{-\alpha}{2}} s - \frac{\beta}{\sqrt{-2\alpha}} \right) & \text{in the case } \sigma(s) = 1 \\ L_\ell^{\beta-1}(-\alpha s) & \text{in the case } \sigma(s) = s \\ P_\ell^{(-(\alpha+\beta)/2-1, (-\alpha+\beta)/2-1)}(s) & \text{in the case } \sigma(s) = 1-s^2 \\ P_\ell^{((\alpha-\beta)/2-1, (\alpha+\beta)/2-1)}(-s) & \text{in the case } \sigma(s) = s^2-1 \\ \left(\frac{s}{\beta}\right)^\ell L_\ell^{1-\alpha-2l}\left(\frac{\beta}{s}\right) & \text{in the case } \sigma(s) = s^2 \\ i^\ell P_\ell^{((\alpha+i\beta)/2-1, (\alpha-i\beta)/2-1)}(is) & \text{in the case } \sigma(s) = s^2+1 \end{cases} \quad (9)$$

where  $H_\ell$ ,  $L_\ell^p$  and  $P_\ell^{(p, q)}$  are the Hermite, Laguerre and Jacobi polynomials, respectively. The relation (9) does not have a very simple form. In certain cases we have to consider

the classical polynomials outside the interval where they are orthogonal or for complex values of parameters.

Let  $\ell \in \mathbb{N}$ ,  $\ell < \Lambda$ , and let  $m \in \{0, 1, \dots, \ell\}$ . The functions

$$\Phi_{\ell,m}^{(\alpha,\beta)}(s) = \kappa^m(s) \frac{d^m}{ds^m} \Phi_{\ell}^{(\alpha,\beta)}(s) \quad \text{where} \quad \kappa(s) = \sqrt{\sigma(s)} \quad (10)$$

are called the *associated special functions*. If we differentiate (5)  $m$  times and then multiply the obtained relation by  $\kappa^m(s)$  then we get the equation

$$\mathcal{H}_m \Phi_{\ell,m}^{(\alpha,\beta)} = \lambda_{\ell} \Phi_{\ell,m}^{(\alpha,\beta)} \quad (11)$$

where  $\mathcal{H}_m$  is the differential operator

$$\begin{aligned} \mathcal{H}_m = & -\sigma(s) \frac{d^2}{ds^2} - \tau(s) \frac{d}{ds} + \frac{m(m-2)}{4} \frac{(\sigma'(s))^2}{\sigma(s)} \\ & + \frac{m\tau(s)}{2} \frac{\sigma'(s)}{\sigma(s)} - \frac{1}{2} m(m-2) \sigma''(s) - m\tau'(s). \end{aligned} \quad (12)$$

For each  $m < \Lambda$ , the special functions  $\Phi_{\ell,m}^{(\alpha,\beta)}$  with  $m \leq \ell < \Lambda$  are orthogonal with respect to the scalar product

$$\langle f, g \rangle = \int_a^b \overline{f(s)} g(s) \varrho(s) ds. \quad (13)$$

The operators  $\mathcal{H}_m$  are directly related to some Schrödinger type operators. If  $(a, b) \longrightarrow (a', b') : s \mapsto x = x(s)$  is a differentiable bijective mapping such that  $dx/ds = \pm 1/\kappa(s)$  and  $(a', b') \longrightarrow (a, b) : x \mapsto s(x)$  is its inverse then the functions

$$\Psi_{\ell,m}^{(\alpha,\beta)}(x) = \sqrt{\kappa(s(x)) \varrho(s(x))} \Phi_{\ell,m}^{(\alpha,\beta)}(s(x)). \quad (14)$$

with  $m \leq \ell < \Lambda$  are orthogonal [2]

$$\int_{a'}^{b'} \overline{\Psi_{\ell,m}^{(\alpha,\beta)}}(x) \Psi_{k,m}^{(\alpha,\beta)}(x) dx = 0 \quad \text{for} \quad \ell \neq k$$

and satisfy the equation [2, 4]

$$\left[ -\frac{d^2}{dx^2} + V_m(x) \right] \Psi_{\ell,m}^{(\alpha,\beta)} = \lambda_{\ell} \Psi_{\ell,m}^{(\alpha,\beta)} \quad (15)$$

where  $V_m(x)$ , defined in terms of the function  $\eta(s) = 1/\sqrt{\kappa(s) \varrho(s)}$ , is given by

$$\begin{aligned} V_m(x) = & \left[ \frac{m(m-2)}{4} \frac{(\sigma'(s))^2}{\sigma(s)} + \frac{m\tau(s)}{2} \frac{\sigma'(s)}{\sigma(s)} - \frac{1}{2} m(m-2) \sigma''(s) \right. \\ & \left. - m\tau'(s) - \sigma(s) \frac{\eta''(s)}{\eta(s)} - \tau(s) \frac{\eta'(s)}{\eta(s)} \right]_{s=s(x)}. \end{aligned} \quad (16)$$

For example, in the case  $\sigma(s) = 1$ , the change of variable  $\mathbb{R} \rightarrow \mathbb{R} : x \mapsto s(x) = x$  leads to

$$V_m(x) = \frac{\alpha^2}{4} x^2 + \frac{\alpha\beta}{2} x + \frac{\beta^2}{4} + \frac{\alpha}{2} - \alpha m. \quad (17)$$

### 3. New explicitly solvable Schrödinger type equations

In the case of a second order differential equation

$$\left[ A(r) \frac{d^2}{dr^2} + B(r) \frac{d}{dr} + C(r) \right] \psi(r) = 0 \quad (18)$$

with  $A(r) \neq 0$  we can eliminate the first order derivative by using the function

$$h(r) = \exp \left( \int^r \frac{B(t)}{2A(t)} dt \right). \quad (19)$$

The equation (18) is equivalent to the equation

$$\frac{1}{A(r)} h(r) \left[ A(r) \frac{d^2}{dr^2} + B(r) \frac{d}{dr} + C(r) \right] \frac{1}{h(r)} h(r) \psi(r) = 0 \quad (20)$$

which can be written as [5, 9]

$$\left[ \frac{d^2}{dr^2} + \frac{4A(r)C(r) - 2A(r)B'(r) + 2B(r)A'(r) - B^2(r)}{4A^2(r)} \right] h(r) \psi(r) = 0. \quad (21)$$

The Schrödinger type equations (15) have the form [4]

$$\left[ -\frac{d^2}{dx^2} + C_1(\alpha, \beta, m) I_1(x) + C_{-1}(\alpha, \beta, m) I_{-1}(x) + C_0(\alpha, \beta, m, \ell) \right] \Psi_{\ell, m}^{(\alpha, \beta)}(x) = 0. \quad (22)$$

If, for  $k \in \{-1, 1\}$ , there exists a differentiable bijective mapping

$$(\tilde{a}, \tilde{b}) \longrightarrow (a', b') : r \mapsto x(r)$$

such that

$$x'(r) = \frac{1}{\sqrt{I_k(x(r))}}$$

then the equation (22) is equivalent to

$$\begin{aligned} & \left[ -\frac{d^2}{dr^2} + C_{-k}(\alpha, \beta, m) \frac{I_{-k}(x(r))}{I_k(x(r))} + \frac{C_0(\alpha, \beta, m, \ell)}{I_k(x(r))} - \frac{5}{16} \frac{(I'_k(x(r)))^2}{(I_k(x(r)))^3} \right. \\ & \left. + \frac{1}{4} \frac{I''_k(x(r))}{(I_k(x(r)))^2} - E \right] \sqrt[4]{I_k(x(r))} \Psi_{\ell, m}^{(\alpha, \beta)}(x(r)) = 0 \end{aligned} \quad (23)$$

where  $E = -C_k(\alpha, \beta, m)$ . We know that  $\alpha, \beta, \ell$  and  $m$  must satisfy certain restrictions.

In [5] the authors consider the Schrödinger equation (translated harmonic oscillator)

$$\left[ -\frac{d^2}{dx^2} + \theta^2 x^2 + \rho x + \lambda \right] \phi(x) = 0. \quad (24)$$

In this case the substitution  $x = \sqrt{2r}$  leads to the equation

$$\left[ -\frac{d^2}{dr^2} + \frac{\rho}{\sqrt{2r}} + \frac{\lambda}{2r} - \frac{3}{16} \frac{1}{r^2} + \theta^2 \right] \sqrt[4]{r} \phi(\sqrt{2r}) = 0. \quad (25)$$

and the substitution  $x = \sqrt[3]{(3r/2)^2}$  to the equation

$$\left[ -\frac{d^2}{dr^2} + \theta^2 \left( \frac{3r}{2} \right)^{\frac{2}{3}} + \lambda \left( \frac{2}{3r} \right)^{\frac{2}{3}} - \frac{5}{36} \frac{1}{r^2} + \rho \right] \sqrt[6]{r} \phi(\sqrt[3]{(3r/2)^2}) = 0. \quad (26)$$

In [5] these equations are considered as two new exactly solvable Schrödinger equations.

This is not obvious because, in equation (24), the eigenvalue  $\lambda$  depends on  $\theta^2$  and  $\rho$ .

The Schrödinger type equation corresponding to the potential (17)

$$\left[ -\frac{d^2}{dx^2} + \frac{\alpha^2}{4} x^2 + \frac{\alpha\beta}{2} x + \frac{\beta^2}{4} + \frac{\alpha}{2} - \alpha m + \alpha \ell \right] \Psi_{\ell, m}^{(\alpha, \beta)}(x) = 0 \quad (27)$$

is satisfied for  $\alpha < 0$  and any  $m, \ell \in \mathbb{Z}$  such that  $0 \leq m \leq \ell$ . It is similar to (24),  $I_1(x) = x^2$ ,  $I_{-1}(x) = x$ ,  $C_1(\alpha, \beta, m) = \frac{\alpha^2}{4}$ ,  $C_{-1}(\alpha, \beta, m) = \frac{\alpha\beta}{2}$  and  $C_0(\alpha, \beta, m, \ell) = \frac{\beta^2}{4} + \frac{\alpha}{2} - \alpha m + \alpha \ell$ . Using the substitution  $x = \sqrt[3]{(3r/2)^2} = \sqrt[3]{\frac{9}{4} r^{\frac{2}{3}}}$  we get the equation

$$\left[ -\frac{d^2}{dr^2} + \frac{\alpha^2}{4} \left(\frac{3r}{2}\right)^{\frac{2}{3}} + \left(\frac{\beta^2}{4} + \frac{\alpha}{2} - \alpha m + \alpha \ell\right) \left(\frac{2}{3r}\right)^{\frac{2}{3}} - \frac{5}{36} \frac{1}{r^2} + \frac{\alpha\beta}{2} \right] \sqrt[6]{r} \Psi_{\ell,m}^{(\alpha,\beta)}(\sqrt[3]{\frac{9}{4} r^{\frac{2}{3}}}) = 0 \quad (28)$$

and by using the substitution  $x = \sqrt{2r}$  the equation

$$\left[ -\frac{d^2}{dr^2} + \frac{\alpha\beta}{2} \frac{1}{\sqrt{2r}} + \left(\frac{\beta^2}{4} + \frac{\alpha}{2} - \alpha m + \alpha \ell\right) \frac{1}{2r} - \frac{3}{16} \frac{1}{r^2} + \frac{\alpha^2}{4} \right] \sqrt[4]{r} \Psi_{\ell,m}^{(\alpha,\beta)}(\sqrt{2r}) = 0. \quad (29)$$

Let  $c_1$  and  $c_2$  be two fixed real numbers. The function

$$\psi_{\ell,m} : (0, \infty) \longrightarrow \mathbb{R}, \quad \psi_{\ell,m}(r) = \sqrt[6]{r} \Psi_{\ell,m}^{(\alpha,\beta)}(\sqrt[3]{\frac{9}{4} r^{\frac{2}{3}}}) \quad (30)$$

is a solution of the equation

$$\left[ -\frac{d^2}{dr^2} + c_1 \left(\frac{3r}{2}\right)^{\frac{2}{3}} + c_2 \left(\frac{2}{3r}\right)^{\frac{2}{3}} - \frac{5}{36} \frac{1}{r^2} \right] \psi = E\psi \quad (31)$$

for  $E = -\frac{\alpha\beta}{2}$  if  $\alpha, \beta, \ell, m$  satisfy the system

$$\begin{cases} \frac{\alpha^2}{4} = c_1 \\ \frac{\beta^2}{4} + \frac{\alpha}{2} - \alpha m + \alpha \ell = c_2 \end{cases} \quad (32)$$

and the conditions  $m, \ell \in \{0, 1, 2, \dots\}$ ,  $m \leq \ell$ ,  $\alpha < 0$ . In the case  $c_1 \geq 0$ , the functions

$$\psi_{\ell,m}^{\pm}(r) = \sqrt[6]{r} \Psi_{\ell,m}^{(-2\sqrt{c_1}, \pm 2\sqrt{c_2 + \sqrt{c_1}(1+2\ell-2m)})}(\sqrt[3]{\frac{9}{4} r^{\frac{2}{3}}}) \quad (33)$$

satisfy the relation

$$\left[ -\frac{d^2}{dr^2} + c_1 \left(\frac{3r}{2}\right)^{\frac{2}{3}} + c_2 \left(\frac{2}{3r}\right)^{\frac{2}{3}} - \frac{5}{36} \frac{1}{r^2} \right] \psi_{\ell,m}^{\pm} = E_{\ell,m}^{\pm} \psi_{\ell,m}^{\pm} \quad (34)$$

for

$$E_{\ell,m}^{\pm} = \pm 2\sqrt{c_1 c_2 + c_1 \sqrt{c_1}(1+2\ell-2m)} \quad (35)$$

if  $m, \ell \in \{0, 1, 2, \dots\}$  are such that

$$m \leq \ell \quad \text{and} \quad c_2 + \sqrt{c_1}(1+2\ell-2m) \geq 0. \quad (36)$$

By using (9), (10), (14) and table 1, the solutions  $\psi_{\ell,m}^{\pm}$  can be expressed in terms of Hermite polynomials

$$\begin{aligned} \psi_{\ell,m}^{\pm}(r) &= \sqrt[6]{r} \exp \left( -\frac{3}{4} \sqrt[3]{\frac{3}{2} \sqrt{c_1}} r^{\frac{4}{3}} \pm \sqrt[3]{\frac{9}{4} \sqrt{c_2 + \sqrt{c_1}(1+2\ell-2m)}} r^{\frac{2}{3}} \right) \\ &\quad \times \left[ \frac{d^m}{dx^m} H_{\ell} \left( \sqrt[4]{c_1} x \mp \frac{1}{\sqrt[4]{c_1}} \sqrt{c_2 + \sqrt{c_1}(1+2\ell-2m)} \right) \right]_{x=\sqrt[3]{\frac{9}{4} r^{\frac{2}{3}}}} \end{aligned} \quad (37)$$

and one can remark that they are square integrable on  $(0, \infty)$ . In view of the well-known relation  $H'_{\ell} = 2\ell H_{\ell-1}$ , the function  $\psi_{\ell,m}^{\pm}(r)$  coincides up to a multiplicative constant to the function

$$\begin{aligned} \psi_n^{\pm}(r) &= \sqrt[6]{r} \exp \left( -\frac{3}{4} \sqrt[3]{\frac{3}{2} \sqrt{c_1}} r^{\frac{4}{3}} \pm \sqrt[3]{\frac{9}{4} \sqrt{c_2 + \sqrt{c_1}(1+2n)}} r^{\frac{2}{3}} \right) \\ &\quad \times \left[ H_n \left( \sqrt[4]{c_1} x \mp \frac{1}{\sqrt[4]{c_1}} \sqrt{c_2 + \sqrt{c_1}(1+2n)} \right) \right]_{x=\sqrt[3]{\frac{9}{4} r^{\frac{2}{3}}}} \end{aligned} \quad (38)$$

where  $n = \ell - m$ . For  $c_1 \geq 0$ , the function  $\psi_n^\pm(r)$  is an eigenfunction of the Schrödinger type operator

$$\mathcal{H} = -\frac{d^2}{dr^2} + c_1 \left(\frac{3r}{2}\right)^{\frac{2}{3}} + c_2 \left(\frac{2}{3r}\right)^{\frac{2}{3}} - \frac{5}{36} \frac{1}{r^2} \quad (39)$$

corresponding to the eigenvalue

$$E_n^\pm = \pm 2\sqrt{c_1 c_2 + c_1 \sqrt{c_1}(1 + 2n)} \quad (40)$$

for any  $n \in \{0, 1, 2, \dots\}$  satisfying the relation

$$n \geq -\frac{c_2}{2\sqrt{c_1}} - \frac{1}{2}. \quad (41)$$

Certain solutions of the equation (see (29))

$$\left[-\frac{d^2}{dr^2} + c_1 \frac{1}{\sqrt{2r}} + c_2 \frac{1}{2r} - \frac{3}{16} \frac{1}{r^2}\right] \psi = E\psi \quad (42)$$

may be found by looking for solutions of the system of equations

$$\begin{cases} \frac{\alpha\beta}{2} = c_1 \\ \frac{\beta^2}{4} + \frac{\alpha}{2} - \alpha m + \alpha \ell = c_2 \end{cases} \quad (43)$$

satisfying the conditions  $\alpha < 0$  and  $m, \ell \in \{0, 1, 2, \dots\}$  with  $0 \leq m \leq \ell$ . The corresponding value of  $E$  is  $-\frac{\alpha^2}{4}$ . The solution of the system (43) leads to an equation of third degree, the formulas are more complicated and will not be presented here. The other cases presented in table 1 may also lead to some new explicitly solvable systems.

#### 4. Concluding remarks

The method proposed by Dereziński and Wrochna allows us to generate new explicitly solvable systems if we take into consideration the parameter dependence of the eigenvalues of the starting system. Some interesting explicitly solvable systems can be obtained by starting from solvable hypergeometric type equations containing several parameters.

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